A NOTE ON SYMMETRIC DIAGONAL EQUATIONS

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1. Introduction

The object of this note is to establish an estimate for the number of solutions of certain simultaneous symmetric diagonal equations which is, in some sense, close to best possible. We shall suppose in what follows that s and t are positive integers, that k_j $(1 \le j \le t)$ are positive integers with $1 \le k_1 < k_2 < \cdots < k_t$, and that P is a large positive real number. Define $S_s(P;\mathbf{k})$ to be the number of solutions of the simultaneous diophantine equations

$$\sum_{i=1}^{s}(x_i^{k_j} - y_i^{k_j}) = 0 \quad (1 \le j \le t) \tag{1}$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le s)$. Estimates for $S_s(P;\mathbf{k})$, for various choices of $\mathbf{k}$, are of importance in the study of Waring's problem, and of simultaneous additive equations. We prove the following estimate.

Theorem 1. *Suppose that $k_t > 1$. Then there is a positive number $A = A(t,\mathbf{k})$ such that*

$$S_{t+1}(P;k_1,\dots,k_t) \ll_{t,\mathbf{k}} P^{t+1}(\log P)^A.$$

Here and throughout $\ll$ denotes Vinogradov's well-known notation. Except for the power of $\log P$ present, Theorem 1 is best possible, for it is not difficult to establish the following lower bound.

Theorem 2. *There is a positive constant B such that*

$$S_s(P;k_1,\dots,k_t) \gg e^{-Bs^2/P}s!P^s + (2s)^{-t}P^{2s-(k_1+\cdots+k_t)}.$$

For $1 \le s \le t$ it is easy to show that $S_s(P;k_1,\dots,k_t) \le k_1\dots k_sP^s$. Meanwhile, for $s > t$ we are at present unable to establish an upper bound of the expected order of magnitude for $S_s(P;\mathbf{k})$ until s is of order $k_t^2 \log k_t$. However, it is widely believed that the lower bound of Theorem 2 is close to the truth.

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Conjecture. *Let $k_t > 2$. Then we have for each s and $\mathbf{k}$,*

$$S_s(P; k_1, \ldots, k_t) \ll_{s,\mathbf{k}} P^s + P^{2s-(k_1+\cdots+k_t)}.$$

When $k_t = 2$ the behaviour of $S_s(P; \mathbf{k})$ is well understood. In this case the conjecture holds except in the exceptional cases $S_2(P; 2) \asymp P^2 \log P$ and $S_3(P; 1, 2) \asymp P^3 \log P$, the factor of $\log P$ arising from the "major arcs" of the Hardy-Littlewood dissection. (No such phenomenon exists for larger k_t). When $s > t+1$, present estimates for $S_s(P; \mathbf{k})$ stem from Vinogradov's mean value theorem, or from analogues of Hua's inequality (see Hua [2] for the methods involved, although for general $\mathbf{k}$ there appears to be nothing in the literature). These estimates miss the expected exponent for P by some margin, whereas Theorem 1 misses the expected result only in the power of $\log P$. Some form of Theorem 1 has already been established in the cases $t = 1$ (classical), $t = 2$ (see Theorem 3 of Wooley [6]), and $\mathbf{k} = (1, 2, \ldots, t)$ and $(1, 2, \ldots, t-1, t+1)$ (see Lemmata 5.2 and 5.4 of Hua [2]). In the last two cases one makes use of Newton's formulae on the roots of polynomials to establish convenient identities. In the first case more is known. We have $S_2(P; k) \ll_{t,k} P^2$ when k is odd (see Hooley [1], and also section 4 of Vaughan [5]). The proof of our results is entirely elementary, making use of divisor sums via the use of a suitable identity. In section 2 we establish some preliminary lemmata required for the proof of Theorem 1 in section 3. In section 4 we establish Theorem 2 by a well-known argument.

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2. Preliminary lemmata

With the notation of the previous section, define polynomials $\phi_{i,s} \in \mathbb{Z}[x_1, \ldots, x_s]$ $(1 \leq i \leq t)$ by

$$\phi_{i,s}(\mathbf{x}) = x_1^{k_i} + \cdots + x_s^{k_i}. \tag{2}$$

We aim to establish a suitable identity with which to make use of divisor sums. The following lemma suffices.

Lemma 1. *There exists a non-trivial polynomial $\Psi \in \mathbb{Z}[y_1, \ldots, y_t]$ such that*

$$\Psi(\phi_{1,t-1}, \ldots, \phi_{t,t-1})$$

is identically zero.

Proof. The result follows immediately by considering transcendence degrees. We let $K = \mathbb{Q}(\phi_{1,t-1}, \ldots, \phi_{t,t-1})$. Then $K \subseteq \mathbb{Q}(x_1, \ldots, x_{t-1})$, so that K has transcendence degree at most $t-1$ over $\mathbb{Q}$. But then the t polynomials $\phi_{i,t-1} \in K$ $(1 \leq i \leq t)$ cannot be algebraically independent, and the existence of the required polynomial Ψ then follows.

Lemma 2. *Let $\Upsilon \in \mathbb{Z}[x_1, \ldots, x_s]$ be a non-trivial homogeneous polynomial of degree k. Then the number of solutions of the diophantine equation $\Upsilon(x_1, \ldots, x_s) = 0$ with $|x_i| \le P$ $(1 \le i \le s)$ is at most kP^{s-1}.*

Proof. We proceed inductively. The result plainly holds when $s = 1$. Suppose that the result holds for each $1 \le s' < s$, and let $\Psi \in \mathbb{Z}[x_1, \ldots, x_s]$ be a non-trivial homogeneous polynomial of degree k. By a rearrangement of variables we have that Ψ is explicit in x_s. Let the degree of Ψ with respect to x_s be r, and write the coefficient of x_s^r as $\Phi(x_1, \ldots, x_{s-1})$. Then Φ is a non-trivial homogeneous polynomial in $s-1$ variables of degree $k-r$. By the inductive hypothesis, the number of solutions of the diophantine equation $\Phi(x_1, \ldots, x_{s-1}) = 0$ with $|x_i| \le P$ $(1 \le i \le s-1)$ is at most $(k-r)P^{s-2}$. Then the number of solutions $(x_1, \ldots, x_s)$ of $\Psi(\mathbf{x}) = 0$ satisfying $\Phi(x_1, \ldots, x_{s-1}) = 0$ and with $|x_i| \le P$ $(1 \le i \le s)$ is at most $(k-r)P^{s-1}$. Meanwhile, if $\Phi(x_1, \ldots, x_{s-1})$ is non-zero then x_s satisfies a non-trivial polynomial of degree r. So there are at most rP^{s-1} solutions with $\Phi(\mathbf{x})$ non-zero. Thus there are at most kP^{s-1} solutions altogether, and the inductive hypothesis holds with $s+1$ replacing s. This completes the proof of the lemma.

We require some information about the determinant $\det(x_j^{k_i-1})_{1 \le i,j \le t}$. This is in fact contained in problem 48 of Chapter 1 (part V) of Pólya and Szegö [4], which states that if $\nu_1, \ldots, \nu_n$ are integers with $0 \le \nu_1 < \nu_2 < \cdots < \nu_n$, and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$, then $\det(\alpha_i^{\nu_j})_{1 \le i,j \le n} > 0$. The following gives greater insight.

Lemma 3. *We have $\det(x_j^{k_i-1})_{1 \le i,j \le t} = K(\mathbf{x}) \prod_{i<j}(x_i - x_j)$, where $K(\mathbf{x})$ is a polynomial in $\mathbb{Z}[x_1, \ldots, x_t]$ taking positive values whenever each of $x_1, \ldots, x_t$ are positive.*

Proof. The lemma follows easily, except for the statement concerning the positivity of the polynomial $K(\mathbf{x})$. The latter follows by using the theory of symmetric functions (in particular, Schur functions), an introduction to which may be found in Macdonald [3]. For the sake of conciseness, we shall adopt the notation used in the latter. By equation (3.1) of Macdonald [3], Chapter I, we have $K(\mathbf{x}) = s_\lambda$, with λ the partition $(k_t - t, k_{t-1} - (t-1), \ldots, k_1 - 1)$. But by equation (5.12) of Chapter I of Macdonald [3], we have $s_\lambda = \sum_T x^T$, where the summation is over all tableaux T of shape λ, and here if the weight of T is $\alpha = (\alpha_1, \ldots, \alpha_t)$, then x^T is the monomial $x_1^{\alpha_1} \ldots x_t^{\alpha_t}$. (A tableau is an entity associated with a partition of an integer; see Macdonald [3] for further details). In particular, this sum is non-empty, and each x^T takes positive values whenever $x_1, \ldots, x_t$ are each positive. It therefore follows that when $x_1, \ldots, x_t$ are each positive, then $K(\mathbf{x})$ is also positive, and this completes the proof of the lemma.

3. The proof of Theorem 1

Throughout this section implicit constants will depend at most on $\mathbf{k}$ and t. By equation (1), we have that $S_{t+1}(P; \mathbf{k})$ is the number of solutions of the simultaneous diophantine equations

$$\sum_{i=1}^t x_i^{k_j} - x_{t+1}^{k_j} = \sum_{i=1}^t y_i^{k_j} - y_{t+1}^{k_j} \quad (1 \le j \le t),$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le t+1)$. On recalling (2), by Lemma 1 there exists a non-trivial polynomial $\Psi \in \mathbb{Z}[y_1, \dots, y_t]$ such that $\Psi(\phi_{1,t} - x_{t+1}^{k_1}, \dots, \phi_{t,t} - x_{t+1}^{k_t})$ is divisible by $\prod_{i=1}^t (x_i - x_{t+1})$. Suppose that the quotient polynomial is $\Phi(\mathbf{x})$. Then for each solution $\mathbf{x}, \mathbf{y}$ counted by $S_{t+1}(P; \mathbf{k})$ we have

$$\Phi(\mathbf{x}) \prod_{i=1}^t (x_i - x_{t+1}) = \Phi(\mathbf{y}) \prod_{i=1}^t (y_i - y_{t+1}). \tag{3}$$

Consider first solutions counted by $S_{t+1}(P; \mathbf{k})$ in which the right hand side of (3) is zero. By Lemma 2, there are $O(P^t)$ choices of $\mathbf{y}$ satisfying the latter condition. Fix any one such choice, and pick any of the P possible choices for x_{t+1}. Then for some fixed integers $N_1, \dots, N_t$ we have that $(x_1, \dots, x_t)$ is a solution of the simultaneous equations

$$\sum_{i=1}^t x_i^{k_j} = N_j \quad (1 \le j \le t). \tag{4}$$

Consider solutions of (4) in which, by a rearrangement of variables, the x_i are equal for $i \in I_m$ $(m = 1, 2, \dots, M)$, with $I_1 = \{1, \dots, i_1\}, I_2 = \{i_1 + 1, \dots, i_2\}, \dots, I_M = \{i_{M-1} + 1, \dots, t\}$, but with the x_i otherwise unequal. Then (4) reduces to

$$\sum_{m=1}^M i_m x_m^{k_j} = N_j \quad (1 \le j \le t). \tag{5}$$

But the determinant

$$\det \left(k_j i_m x_m^{k_j - 1} \right)_{1 \le m, j \le M}$$

is non-zero, by Lemma 3, so by the Implicit Function Theorem there are $O(1)$ solutions to (5). Then the total number of solutions to (4) is also $O(1)$, and hence the number of solutions counted by $S_{t+1}(P; \mathbf{k})$ in which the right hand side of (3) is zero is $\ll P^{t+1}$.

Now write $\Upsilon(\mathbf{x})$ for $\Phi(\mathbf{x}) \prod_{i=1}^t (x_i - x_{t+1})$, and consider any solution $\mathbf{x}, \mathbf{y}$ counted by $S_{t+1}(P; \mathbf{k})$ for which $\Upsilon(\mathbf{y}) \neq 0$. Now, if $d(n)$ denotes the divisor function, then there are at most $\left(d\left(|\Upsilon(\mathbf{y})| \right) \right)^t$ possible choices for $x_i - x_{t+1}$ $(1 \le i \le t)$. Fix any one such, say $x_i = x_{t+1} + d_i$ $(1 \le i \le t)$. Then on substitution, for some fixed integer $N = N(\mathbf{y})$, we have that x_{t+1} satisfies the non-trivial equation

$$\sum_{i=1}^t (x_{t+1} + d_i)^{k_1} - x_{t+1}^{k_1} = N.$$

There are therefore $O(1)$ solutions possible for x_{t+1}, and so the number of solutions of this type counted by $S_{t+1}(P;\mathbf{k})$ is

$$\ll \sum_{\mathbf{y}}{}^{*} \left(d\left(|\Upsilon(\mathbf{y})|\right)\right)^{t}. \tag{6}$$

Here the summation is over $\mathbf{y}$ with $1 \le y_i \le P$ $(1 \le i \le t+1)$ satisfying $\Upsilon(\mathbf{y}) \ne 0$. But by Theorem 3 of Hua [2], the sum in (6) is $\ll_{t,\mathbf{k}} P^{t+1}(\log P)^A$ for some $A = A(t,\mathbf{k})$. This completes the proof of the theorem.

4. A lower bound

We establish Theorem 2 in two stages.

First suppose that $[P] \ge s$. Then by taking all permutations of the solutions with $x_i = y_i$ $(1 \le i \le s)$ and $x_i \ne x_j$ $(i \ne j)$ in (1), we have $S_s(P;\mathbf{k}) \gg s![P]!/[P-s]!$. It is a straightforward exercise in the use of Stirling's formula to show that the latter is $\gg \exp(-Bs^2/P)s!P^s$ for some absolute constant B. Now suppose that $s/\log s < [P] < s$. Then by considering solutions in blocks of $[P]$ elements with $x_i \ne x_j$ for $i \ne j$ in each block, and permutations thereof, we find that $S_s(P;\mathbf{k}) \gg (([P]!)^{s/P})^2$. Again, by using Stirling's formula we find that when $[P] \ge s/\log s$ the latter is $\gg \exp(-Bs^2/P)s!P^s$. Finally, when $[P] \le s/\log s$, it suffices to use the trivial bound $S_s(P;\mathbf{k}) \gg [P]^s$.

Next consider $S_s(P;\mathbf{k};\mathbf{h})$, which we define to be the number of solutions of the simultaneous equations

$$\sum_{i=1}^{s}(x_i^{k_j} - y_i^{k_j}) = h_j \quad (1 \le j \le t)$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le s)$. We have

$$\sum_{\mathbf{h}} S_s(P;\mathbf{k};\mathbf{h}) = [P]^{2s}$$

where the summation is over $\mathbf{h}$ with $|h_j| \le s(P^{k_j}-1)$ $(1 \le j \le t)$. Let $e(\alpha)$ denote $e^{2\pi i\alpha}$, and define $f(\boldsymbol{\alpha}) = \sum_{x\le P} e(\alpha_1 x^{k_1} + \cdots + \alpha_t x^{k_t})$. Then

$$S_s(P;\mathbf{k};\mathbf{h}) = \int_{\mathbb{T}^t} |f(\boldsymbol{\alpha})|^{2s} e(-\boldsymbol{\alpha}.\mathbf{h})d\boldsymbol{\alpha},$$

where $\mathbb{T}^t$ denotes the t-dimensional unit cube. Then

$$\sum_{\mathbf{h}} \int_{\mathbb{T}^t} |f(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} \ge \sum_{\mathbf{h}} \int_{\mathbb{T}^t} |f(\boldsymbol{\alpha})|^{2s} e(-\boldsymbol{\alpha}.\mathbf{h})d\boldsymbol{\alpha} \gg P^{2s},$$

and hence

$$S_s(P;\mathbf{k}) \gg P^{2s}\left(\sum_{\mathbf{h}} 1\right)^{-1} \gg (2s)^{-t}P^{2s-(k_1+\cdots+k_t)},$$

which completes the proof of the theorem.

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